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HOLOMORPHIC MORSE INEQUALITIES AND SYMPLECTIC REDUCTION

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We introduce Morse-type inequalities for a holomorphic circle action on a holomorphic vector bundle over a compact Kähler manifold. Our inequalities produce bounds on the multiplicities of weights occurring in the twisted Dolbeault cohomology in terms of the data of the fixed points and of the symplectic reduction. This result generalizes both the Wu–Zhang extension of Witten's holomorphic Morse inequalities and the Tian–Zhang Morse-type inequalities for symplectic reduction. As an application we get a new proof of the Tian–Zhang relative index theorem for symplectic quotients.

1. INTRODUCTION

Let (M, ω) be a compact Kähler manifold of complex dimension n and let E be a holomorphic Hermitian vector bundle over M with the compatible holomorphic connection. We denote by $H^*(M, \mathcal{O}(E))$ the cohomology groups with coefficients in the sheaf of holomorphic sections of E, calculated from the twisted Dolbeault complex $(\Omega^{0,*}(M, E), \bar{\partial}_E)$.

Suppose that the circle group S^1 acts holomorphically and effectively on M preserving the Kähler structure and that the action is lifted to E preserving the Hermitian structure on E. Then, for each $p = 0, 1, \ldots, n$, we obtain a representation of S^1 on $H^p(M, \mathcal{O}(E))$. The holomorphic inequalities we are going to discuss in this paper give an estimate on the characters of these representations.

More precisely, assume that $\mu: M \to \mathbb{R}$ is a momentum map for the circle action on M and that $a \in \mathbb{R}$ is a regular value of μ. Our main result is Theorem 2.4 which estimate the character of $H^p(M, \mathcal{O}(E))$ in terms of the fixed-point data and the structure of the *reduced space* $M_a = \mu^{-1}(a)/S^1$. This result unifies and generalizes two known estimates of the twisted Dolbeault cohomology; one in terms of fixed point data only [15, see (2.11)] and another in terms of the reduced space only [11, see (2.14)] (note, however, that in [11] the inequalities are obtained for a much more general case where the circle is replaced by an arbitrary compact Lie group).

The proof of our main theorem is based on the Witten deformation of the Dolbeault operator $\bar{\partial}$. Technically it is very simple since all the necessary calculations are already contained in [15, 11]. Moreover, we need only a very simple version of the calculations in [11] since we work with a circle and not with an arbitrary compact Lie group.

As an application of our inequalities we get a new equivariant index formula (2.17). This formula unifies and generalizes two classical results: the Atiyah–Bott–Segal–Singer fixed point formula for the equivariant index ([1, 2, 4], see also [5, Theorem 6.16]) on one hand, and Guillemin and Sternberg's "Kähler quantization commutes with reduction" theorem ([9, Theorem 5.2], see also (2.18)) on the other hand.

Note, that though our inequalities (2.8) make sense only for Kähler manifolds, the index formula (2.17) holds for arbitrary symplectic manifolds.

Using our index formula (2.17) we obtain a new proof of the Tian–Zhang relative index theorem for symplectic quotients [12, Theorem 5.7].

2. MAIN RESULTS

In this section we formulate our main result (Theorem 2.4) and discuss various applications. The proof of Theorem 2.4 is postponed to the next section.

2.1. Weights and formal characters

Irreducible representation of the circle group $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$ are classified by integer *weights* (here we use the identification of the Lie algebra of S^1 with $\mathbb{R}$ which takes the *negative* primitive lattice element, $-2\pi i \in i\mathbb{R} = Lie(S^1)$, to 1). A representation of weight $k \in \mathbb{Z}$ is isomorphic to the complex line $\mathbb{C}$ on which the element $e^{i\theta} \in S^1$ acts by multiplication by $e^{-ik\theta}$.

If W is a finite dimensional representation of S^1 we denote by $\mathrm{mult}_k(W)$ the multiplicity of the weight $k \in \mathbb{Z}$ in W. Note that the multiplicity $\mathrm{mult}_0(W)$ of the zero weight is equal to the dimension of the space W^{S^1} of vectors in W which are invariant with respect to the action of S^1. Let *support* of W be $\mathrm{supp}(W) = \{k \in \mathbb{Z} : \mathrm{mult}_k(W) \neq 0\}$.

The *formal character* of W is the formal sum

$$\mathrm{char}(W) = \sum_{k\in\mathbb{Z}} \mathrm{mult}_k(W)\, e^{-ik\theta}.$$

It lies in the ring $\mathbb{Z}[e^{i\theta}, e^{-i\theta}]$ of Laurent polynomials in $e^{i\theta}$ with integer coefficients. This ring is called the *ring of formal characters* of the circle group. We will also consider the completion $\mathscr{L} = \mathbb{Z}[[e^{i\theta}, e^{-i\theta}]]$ of this ring. The elements of $\mathscr{L}$ are formal infinite sums $q(\theta) = \sum_{k\in\mathbb{Z}} q_k e^{-ik\theta}$ where $q_k \in \mathbb{Z}$.

2.2. Momentum map and symplectic reduction

Let V denote the vector field on M that generates the S^1-action. We will assume that S^1-action is *Hamiltonian*, i.e. there is a moment map $\mu : M \to \mathbb{R}$ such that $\iota_V\omega = d\mu$. Note [8] that it is always the case if the fixed-point set of S^1 on M is non-empty.

Assume that $a \in \mathbb{R}$ is a regular value of the momentum map μ. Then $\mu^{-1}(a) \subset M$ is a smooth submanifold endowed with a locally free action of S^1. For simplicity, we will assume that this action is free. Then the quotient space $M_a = \mu^{-1}(a)/S^1$ is a smooth manifold. Moreover [9], M_a inherits a Kähler structure from M and the vector bundle E descends to a holomorphic Hermitian vector bundle E_a over M_a.

Let $\mathscr{F} = M \times \mathbb{C}$ denote the trivial line bundle over M with the S^1 action defined by the formula

$$e^{i\theta} : (x, z) \mapsto (e^{i\theta} \cdot x, e^{i\theta} z), \qquad x \in M, z \in \mathbb{C}. \tag{2.1}$$

Denote by $q : \mu^{-1}(a) \to M_a = \mu^{-1}(a)/S^1$ the projection. The restriction $\mathscr{F}|_{\mu^{-1}(a)}$ of $\mathscr{F}$ on $\mu^{-1}(a)$ descends to a unique bundle $\mathscr{F}_a$ over M_a such that $q^*\mathscr{F}_a = \mathscr{F}|_{\mu^{-1}(a)}$.

Let $\mathscr{F}^{-1}, \mathscr{F}_a^{-1}$ be the dual bundles to $\mathscr{F}$ and $\mathscr{F}_a$, respectively. If $k \geqslant 0$, we denote by $\mathscr{F}^{\pm k}$ (resp. $\mathscr{F}_a^{\pm k}$) the kth tensor power of the bundle $\mathscr{F}^{\pm 1}$ (resp. $\mathscr{F}_a^{\pm 1}$). Note that

$q^*\mathscr{F}_a^k = \mathscr{F}^k$. Obviously,

$$\mathrm{mult}_m H^*(M, \mathscr{O}(E \otimes \mathscr{F}^k)) = \mathrm{mult}_{m+k} H^*(M, \mathscr{O}(E)) \tag{2.2}$$

for any $k, m \in \mathbb{Z}$.

We will be interested in the cohomology $H^*(M_a, \mathscr{O}(E_a \otimes \mathscr{F}_a^k))$ of M_a with coefficients in the sheaf of holomorphic sections of $E_a \otimes \mathscr{F}_a^k$.

There is another action of S^1 on $\mathscr{F}$ given by the formula

$$\mathrm{e}^{\mathrm{i}\theta}: (x, z) \mapsto (x, \mathrm{e}^{\mathrm{i}\theta} z), \quad x \in M, \quad z \in \mathbb{C}. \tag{2.3}$$

This action commutes with (2.1) and, hence, reduces to $\mathscr{F}_a$. Therefore we obtain induced actions of S^1 on $\mathscr{F}_a^k$. This action preserve the base points in $\mathscr{F}$ and the weight of the representation on the fibers is equal to $-k$.

Let $S(\mathscr{F}_a, \mathscr{F}_a^{-1})$ denote the direct sum

$$S(\mathscr{F}_a, \mathscr{F}_a^{-1}) = \bigoplus_{k \in \mathbb{Z}} \mathscr{F}_a^k.$$

One should think about $S(\mathscr{F}_a, \mathscr{F}_a^{-1})$ as the symmetric algebra in $\mathscr{F}_a, \mathscr{F}_a^{-1}$. This is an infinite-dimensional S^1-equivariant bundle over M_a. However the cohomology groups $H^*(M_a, \mathscr{O}(E_a \otimes S(\mathscr{F}_a, \mathscr{F}_a^{-1})))$ are the sums of those with coefficients in the line bundles $\mathscr{F}_a^k$. Therefore the multiplicity of any weight k in $H^*(M_a, \mathscr{O}(E_a \otimes S(\mathscr{F}_a, \mathscr{F}_a^{-1})))$ is finite and the character

$$\mathrm{char}\, H^*(M_a, \mathscr{O}(E_a \otimes S(\mathscr{F}_a, \mathscr{F}_a^{-1}))) = \sum_{k \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}k\theta} \dim_{\mathbb{C}} H^*(M_a, \mathscr{O}(E_a \otimes \mathscr{F}_a^k)) \tag{2.4}$$

is well defined as an element of $\mathscr{L}$.

2.3. Fixed-point set

Suppose that F is a connected component of the fixed-point set of the S^1-action on M. Then F is a compact Kähler manifold. Let $n - n_F$ be the complex dimension of F. The complexification of the normal bundle $N_F \to F$ in M has the decomposition $N_F^{\mathbb{C}} = N_F^{(1,0)} \oplus N_F^{(0,1)}$, where $N_F^{(1,0)}$ is a holomorphic vector bundle over F of rank n_F. The circle S^1 acts on N_F preserving the base points in F. Moreover, the weights of the isotropy representations on the normal fiber are constant along F.

Let λ_k $(1 \leqslant k \leqslant n_F)$ be the isotropy weights on $N_F^{(1,0)}$. Let $N_F^{\pm,(1,0)}$ be the direct sum of the sub-bundles corresponding to the weights $\lambda_k > 0$ and $\lambda_k < 0$, respectively. We denote by ν_F the rank of the holomorphic vector bundle $N_F^{-,(1,0)}$.

The *polarized symmetric tensor products* (cf. [6, 15]) are the vector bundles

$$\begin{aligned} K_F^+ &= S((N_F^{+,(1,0)})^*) \otimes S(N_F^{-,(1,0)}) \otimes \det(N_F^{-,(1,0)}) \\ K_F^- &= S((N_F^{-,(1,0)})^*) \otimes S(N_F^{+,(1,0)}) \otimes \det(N_F^{+,(1,0)}). \end{aligned} \tag{2.5}$$

Here $S((N_F^{\pm,(1,0)})^*)$, $S(N_F^{\pm,(1,0)})$ denote the sums of all symmetric powers of the bundles $(N_F^{\pm,(1,0)})^*$ and $N_F^{\pm,(1,0)}$, respectively and $\det(N_F^{\pm,(1,0)})$ denotes the top exterior power of $N_F^{\pm,(1,0)}$.

The fiber E_p over each fixed point $p \in F$ is a representation of S^1, and $\mathrm{char}(E_p)$ is independent on $p \in F$. Consider the infinite dimensional holomorphic bundles $K_F^{\pm} \otimes E|_F$. The circle acts on the total space while preserving the base points in F. A sub-bundle of any given weight is a holomorphic vector bundle of finite rank, i.e.,

$$K_F^{\pm} \otimes E|_F = \bigoplus_{k \in \mathbb{Z}} E_{F,k}^{\pm} \tag{2.6}$$

where $E_{F,k}^{\pm}$ is an S^1-invariant sub-bundle of finite rank on which the circle acts with weight k. The cohomology groups $H^*(F, \mathscr{O}(K_F^{\pm} \otimes E|_F))$ are the sums of those with coefficients in $E_{F,k}^{\pm}$, each equipped with an induced S^1-action. Therefore, for any $k \in \mathbb{Z}$ the multiplicities

$$\operatorname{mult}_k H^*(F, \mathscr{O}(K_F^{\pm} \otimes E|_F)) = \dim_{\mathbb{C}} H^*(F, \mathscr{O}(E_{F,k}^{\pm})) \tag{2.7}$$

are finite.

Our main result is the following

THEOREM 2.4. *For any $k \in \mathbb{Z}$ there exists a polynomial $Q_k(t)$ with non-negative integer coefficients such that*

$$\begin{aligned} &\sum_{p=0}^{n-1} t^p \dim_{\mathbb{C}} H^p(M_a, \mathscr{O}(E_a \otimes F_a^k)) + \sum_{\mu|_F > a} t^{n_F - \nu_F} \sum_{p=0}^{n-n_F} t^p \operatorname{mult}_k H^p(F, \mathscr{O}(K_F^- \otimes E|_F)) \\ &\qquad + \sum_{\mu|_F < a} t^{\nu_F} \sum_{p=0}^{n-n_F} t^p \operatorname{mult}_k H^p(F, \mathscr{O}(K_F^+ \otimes E|_F)) \\ &\quad = \sum_{p=0}^{n} t^p \operatorname{mult}_k H^p(M, \mathscr{O}(E)) + (1+t)\, Q_k(t) \end{aligned} \tag{2.8}$$

where the second and third sums on the left hand side are taken over connected components of the fixed-point set.

Theorem 2.4 is proven in Section 3.

The rest of this section is devoted to discussion of different applications and a reformulation of Theorem 2.4.

2.5. Reformulation in the language of characters

It follows from (2.7) that the character

$$\operatorname{char} H^p(F, \mathscr{O}(K_F^{\pm} \otimes E|_F)) = \sum_{k \in \mathbb{Z}} \mathrm{e}^{-ik\theta} \dim_{\mathbb{C}} H^p(F, \mathscr{O}(E_{F,k}^{\pm})) \tag{2.9}$$

of the infinite dimensional representation $H^p(F, \mathscr{O}(K_F^{\pm} \otimes E|_F))$ is well defined as an element of $\mathscr{L}$.

Definition 2.6. Let $q(\theta) = \sum_{k \in \mathbb{Z}} q_k \mathrm{e}^{-ik\theta}$ be a formal character of S^1, we say $q(\theta) \geqslant 0$ if $q_k \geqslant 0$ for all $k \in \mathbb{Z}$. Let $Q(\theta, t) = \sum_{m=0}^{n} q_m(\theta) t^m$ be a polynomial of degree n with coefficients in $\mathscr{L}$, we say $Q(\theta, t) \geqslant 0$ if $q_m(\theta) \geqslant 0$ for all m.

For two such polynomials $P(\theta, t)$ and $Q(\theta, t)$, we say $P(\theta, t) \leqslant Q(\theta, t)$ if $Q(\theta, t) - P(\theta, t) \geqslant 0$.

Using (2.4) we can reformulate Theorem 2.4 in the language of characters.

THEOREM 2.7. *There exists a polynomial $Q(\theta, t) \in \mathscr{L}[t]$, such that $Q \geqslant 0$ and*

$$\sum_{p=0}^{n-1} t^p \operatorname{char} H^p(M_a, \mathcal{O}(E_a \otimes S(\mathscr{F}_a, \mathscr{F}_a^{-1}))) + \sum_{\mu|_F > a} t^{n_F - \nu_F} \sum_{p=0}^{n-n_F} t^p \operatorname{char} H^p(F, \mathcal{O}(K_F^- \otimes E|_F))$$

$$+ \sum_{\mu_F < a} t^{\nu_F} \sum_{p=0}^{n-n_F} t^p \operatorname{char} H^p(F, \mathcal{O}(K_F^+ \otimes E|_F))$$

$$= \sum_{p=0}^{n} t^p \operatorname{char} H^p(M, \mathcal{O}(E)) + (1+t)\, Q(\theta, t). \tag{2.10}$$

2.8. *Witten–Wu–Zhang inequalities*

Theorem 2.7 provides estimates on the character of $H^*(M, \mathcal{O}(E))$ for any regular value $a \in \mathbb{R}$ of the momentum map such that the circle acts freely on $\mu^{-1}(a)$. Let as choose $a < \min\{\mu(x) : x \in M\}$. Then the reduced space M_a is empty and the first and the third summands on the left hand side of (2.10) vanish. Hence, (2.10) reduces to

$$\sum_F t^{n_F - \nu_F} \sum_{p=0}^{n-n_F} t^p \operatorname{char} H^p(F, \mathcal{O}(K_F^- \otimes E|_F)) = \sum_{p=0}^{n} t^p \operatorname{char} H^p(M, \mathcal{O}(E)) + (1+t)\, Q(\theta, t) \tag{2.11}$$

where the sum on the left is taken over all connected components of the fixed-point set. This is precisely the Wu–Zhang extension of the Witten holomorphic Morse inequalities for a circle action [15, Theorem 2.4].

Note that choosing $a > \max\{\mu(x) : x \in M\}$ leads to inequalities which are similar but different from (2.11). It is shown in [14] that combination of those inequalities with (2.11) gives much better estimates than (2.11) alone. Even more information about $H^*(M, \mathcal{O}(E))$ may be obtained by considering (2.10) with all possible values of a.

2.9. *The Tian–Zhang inequalities for symplectic reduction*

Let F be a connected component of the fixed-point set. Given a vector bundle V over F on which the circle acts preserving the base points in F, the fibers of V become representations of the circle which are isomorphic to each other. By the multiplicity of certain weight in V we will understand the multiplicity of this weight in one of the fibers of V. Also we will denote by supp V the set of all weights with non-zero multiplicity.

Let λ_F^+ (resp. λ_F^-) denote the sum of the positive (resp. negative) weights in the representation of the circle on the normal bundle N_F to F. One easily checks that $\operatorname{supp} K_F^+ \subset (-\infty, -|\lambda_F^-|]$ and $\operatorname{supp} K_F^- \subset [\lambda_F^+, \infty)$. It follows that, if $\operatorname{supp} E|_F \subset [k_1, k_2]$ then

$$\operatorname{supp} K_F^+ \otimes E|_F \subset (-\infty, k_2 - |\lambda_F^-|]; \qquad \operatorname{supp} K_F^- \otimes E|_F \subset [k_1 + \lambda_F^+, \infty).$$

Hence,

$$\begin{aligned} &\operatorname{supp} H^p(F, \mathcal{O}(K_F^+ \otimes E|_F)) \subset (-\infty, k_2 - |\lambda_F^-|]; \\ &\operatorname{supp} H^p(F, \mathcal{O}(K_F^- \otimes E|_F)) \subset [k_1 + \lambda_F^+, \infty). \end{aligned} \tag{2.12}$$

If there exists an integer k which is greater than $k_2 - |\lambda_F^-|$ for any F with $\mu(F) < a$ and which is smaller than $k_1 + \lambda_F^+$ for any F with $\mu(F) > a$ then (2.8), (2.12) imply that

$$\sum_{p=0}^{n-1} t^p \dim_{\mathbb{C}} H^p(M_a, \mathcal{O}(E_a \otimes \mathscr{F}_a^k)) = \sum_{p=0}^{n} t^p \operatorname{mult}_k H^p(M, \mathcal{O}(E)) + (1+t)\, Q_k(t). \tag{2.13}$$

(Note that the calculations which led us to (2.13) are similar to those in [7, Section 4].)

Consider the case when E is a *pre-quantum line bundle*. That means that the Kähler form ω represents the Chern class of E in the cohomology $H^2(M)$. For this case there is a natural choice of the momentum map μ given by the Kostant formula: the infinitesimal generator of the action of S^1 on the space of sections of E is given by $-\nabla_V^E + 2\pi i\mu$, where ∇_V^E is the covariant derivative along the vector field V which generates the action of S^1 on M.

With this choice of μ one easily checks that the weight of the representation of S^1 on $E|_F$ is positive (resp. negative) if $\mu(F) > 0$ (resp. $\mu(F) < 0$). It follows that (2.13) holds for $k = 0$, $a = 0$. In other words, there exists a polynomial $Q(t)$ with non-negative coefficients such that

$$\sum_{p=0}^{n-1} t^p \dim_{\mathbb{C}} H^p(M_0, \mathcal{O}(E_0)) = \sum_{p=0}^{n} t^p \dim H^p(M, \mathcal{O}(E))^{S^1} + (1+t)\, Q(t). \tag{2.14}$$

(here $H^p(M, \mathcal{O}(E))^{S_1}$ denotes the space of S^1 invariant vectors in $H^p(M, \mathcal{O}(E))$). Equation (2.14) is precisely the Tian–Zhang Morse-type inequalities for symplectic reduction on a Kähler manifold [11, Theorem 5.1] (note, however, that in [11] the inequalities are obtained for a much more general case where the circle is replaced by an arbitrary compact Lie group).

2.10. Index theorem

An interesting corollary of Theorem 2.4 may be obtained by setting $t = -1$ in (2.8). Then the last summand in the right hand side of (2.8) vanishes and we obtain a combination of the Atiyah–Bott–Segal–Singer fixed point theorem [1, 2, 4] and the Guillemin–Sternberg "quantization commutes with reduction" theorem [9, Theorem 5.2]. We will now explain this in more details.

For a connected component F of the fixed-point set, define

$$\begin{aligned} \operatorname{ind}_k(F; K_F^+ \otimes E|_F) &= \sum_{p=0}^{n-n_F} (-1)^{p+\nu_F} \operatorname{mult}_k H^p(F, \mathcal{O}(K_F^+ \otimes E|_F)) \\ \operatorname{ind}_k(F; K_F^- \otimes E|_F) &= \sum_{p=0}^{n-n_F} (-1)^{p+n_F-\nu_F} \operatorname{mult}_k H^p(F, \mathcal{O}(K_F^- \otimes E|_F)). \end{aligned} \tag{2.15}$$

Recall that the bundles $E_{F,k}^{\pm}$ were introduced in (2.6). By the Riemann–Roch–Hirzebruch theorem [3] (see also [5, Theorem 4.9]) we have

$$\operatorname{ind}_k(F; K_F^{\pm} \otimes E|_F) = \int_F Td(F)\, ch(E_{F,k}^{\pm}), \tag{2.16}$$

where Td and ch stand for the Todd class and Chern character, respectively.

Setting $t = -1$ in (2.8) and taking into account (2.15), we obtain

$$\begin{aligned} &\sum_{p=0}^{n-1} (-1)^p \dim_{\mathbb{C}} H^p(M_a, \mathcal{O}(E_a \otimes F_a^k)) + \sum_{\mu|_F > a} \operatorname{ind}_k(F; K_F^- \otimes E|_F) \\ &\quad + \sum_{\mu|_F < a} \operatorname{ind}_k(F; K_F^+ \otimes E|_F) = \sum_{p=0}^{n} (-1)^p \operatorname{mult}_k H^p(M, \mathcal{O}(E)). \end{aligned} \tag{2.17}$$

Remark 2.11. Assume now that the circle group acts hamiltonially on a symplectic manifold M and that E is an equivariant Hermitian vector bundle over M, endowed with invariant Hermitian connection. Then the individual cohomology $H^p(M, \mathcal{O}(E))$ and $H^p(M_a, \mathcal{O}(E_a \otimes \mathcal{F}_a^k))$ make no sense. However, the alternating sums which appear in (17) may be defined as indexes of corresponding Dirac operators (cf., e.g [11]). Hence, (2.17) makes sense and indeed holds for this, more general, case. This may be shown by a verbatim repetition of the proof of (2.8) (cf. Section 3).

If in (2.17) we choose $a < \min\{\mu(x) : x \in M\}$, then the first and third terms on the left hand side of (2.17) vanish and, in view of (2.16), we get the Atiyah–Bott–Segal–Singer fixed-point theorem. On the other hand, if E is a pre-quantum line bundle $a = 0$ and $k = 0$, then (cf. Section 2.9) the second and the third terms on the left hand side of (2.17) vanish and we obtain the Guillemin-Sternberg "quantization commutes with reduction" theorem [9, Theorem 5.2]:

$$\sum_{p=0}^{n-1} (-1)^p \dim_{\mathbb{C}} H^p(M_0, \mathcal{O}(E_0)) = \sum_{p=0}^{n} (-1)^p \dim_{\mathbb{C}} H^p(M, \mathcal{O}(E))^{S^1}. \tag{2.18}$$

2.12. The Tian–Zhang relative index theorem for symplectic quotients

Let $a < b$ be two regular values of the momentum map μ and suppose that S^1 acts freely on $\mu^{-1}(a)$, $\mu^{-1}(b)$. Then we can form smooth manifolds M_a and M_b as in Section 2.2 and then apply (2.17) to each one of them. Comparing the results we obtain

$$\sum_{p=0}^{n-1} (-1)^p \dim_{\mathbb{C}} H^p(M_b, \mathcal{O}(E_b \otimes \mathcal{F}^k_b)) - \sum_{p=0}^{n-1} (-1)^p \dim_{\mathbb{C}} H^p(M_a, \mathcal{O}(E_a \otimes \mathcal{F}^k_a))$$

$$= \sum_{a < \mu|_F < b} \mathrm{ind}_k(F; K_F^- \otimes E|_F) - \sum_{a < \mu|_F < b} \mathrm{ind}_k(F; K_F^+ \otimes E|_F). \tag{2.19}$$

This formula was first obtained by Tian and Zhang [12, Theorem 5.7] using rather sophisticated study of the spectral flow of a family of Dirac operators with the Atiyah–Patodi–Singer boundary conditions on a symplectic manifold with boundary. (In fact, Tian and Zhang considered only the case $k = 0$. However, (2.19) follows easily from their result).

The result of Tian and Zhang is valid for a more general case where M is an arbitrary symplectic manifold. Note that our proof may be easily extended to that case (cf. Remark 2.11).

3. PROOF OF THEOREM 2.4

First of all, note that it is enough to prove Theorem 2.4 for $k = 0$. Indeed, suppose that the theorem is proven for $k = 0$ and recall that bundles $\mathcal{F}^k$ are defined in Section 2.2. Applying Theorem 2.4 with $k = 0$ to the tensor product $E \otimes \mathcal{F}^m$ and using (2.2) we obtain the statement of the theorem for $k = m$.

Let us prove Theorem 2.4 for $k = 0$. In other words, we will be interested in S^1-invariant elements of $H^*(M, \mathcal{O}(E))$ and $H^*(F, \mathcal{O}(K_F^{\pm} \otimes E|_F))$. Also, without loss of generality, we assume that $a = 0$.

Recall that $\mu : M \to \mathbb{R}$ is a momentum map for the circle action on M. Following [11], we consider a one parameter family of differentials $\bar{\partial}_t : \Omega^{0,*}(M, E) \to \Omega^{0,*+1}(M, E)$ defined by

$$\bar{\partial}_t \alpha = e^{-t|\mu|^2} \bar{\partial} e^{t|\mu|^2} \alpha = \bar{\partial}\alpha + 2t\mu\bar{\partial}\mu \wedge \alpha.$$

Let $\bar{\partial}_t^*$ denote the formal adjoint to $\bar{\partial}_t$ and consider the corresponding Laplacian

$$\bar{\square}_t = \bar{\partial}_t^* \bar{\partial}_t + \bar{\partial}_t \bar{\partial}_t^*.$$

Clearly, for each $t \in \mathbb{R}$ the cohomology $H^*(M, \mathcal{O}(E))$ is isomorphic to the kernel $\mathrm{Ker}\bar{\square}_t$ of $\bar{\square}_t$. Moreover, the S^1 invariant part of $H^*(M, \mathcal{O}(E))$ is isomorphic to the kernel of the restriction of $\bar{\square}_t$ on the space $(\Omega^{0,*}(M,E))^{S^1}$ of S^1-invariant anti-holomorphic differential forms. The later operator is calculated in [11]. It is shown in [11] that, for $t \to \infty$, the calculation of the kernel may be localized to small neighborhoods of $\mu^{-1}(0)$ and of the fixed-point set of the action of S^1. Such a localization, by standard techniques of [11, 13–15], leads to Morse-type inequalities. The contribution of $\mu^{-1}(0)$ to these inequalities is calculated in [11] and is precisely equal to the first summand in the left-hand side of (2.8).

The contribution of the fixed-point set to the inequalities may be calculated using the technique of [15]. Let F be a connected component of the fixed-point set. Then the restriction of μ to F is a constant. Moreover, $\mu(F) \neq 0$ since 0 is a regular value of μ.

In [15], Zhang and Wu considered a one parameter deformation of $\bar{\partial}$ given by

$$\bar{\partial}'_s \alpha = \bar{\partial}\alpha + s\bar{\partial}\mu \wedge \alpha.$$

Near F our operator $\bar{\partial}_t$ looks like $\bar{\partial}'_s$ with $s = \mu(F)t$. Hence, if $\mu(F) > 0$, the asymptotic behavior for $t \to \infty$ of the eigenforms of $\bar{\square}_t$ which concentrate near F is the same as in [15]. In particular, the contribution of F to the inequalities is the same as in [15]. This leads to the second summand on the left hand side of (2.8).

If $\mu(F) < 0$, then, as $t \to \infty$, the operator $\bar{\square}_t$ behaves as the corresponding operator in [15] behaves for $s \to -\infty$. This leads to the last term on the left hand side of (2.8).

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